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HEAT TRANSFER IN ENTRY LENGTH OF DOUBLE PIPES

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Abstract—The velocity distribution, the pressure drop, and the length of the hydrodynamic entry length of the pipes with annular space (annulus) are found theoretically by the hydrodynamics, Bessel functions and the finite Hankel transform, and the examples of numerical calculations are also shown.

The theoretical researches of laminar heat transfer of double pipes in hydrodynamic entry length and in thermal entry length are accomplished. In the theoretical solutions, non-linear Volterra's integral equation and Gauss' method with a high accuracy in the method of numerical integration, are used and it is described that theoretical solutions coincide with experiments very well. Moreover, theoretical calculations of length of hydrodynamic entry region and thermal entry region of double pipes are described.

Résumé—La théorie hydrodynamique permet, à l'aide des fonctions de Bessel et de la transformation de Hankel, le calcul de la distribution des vitesses, la chute de pression et la longueur d'entrée hydrodynamique des conduites annulaires (tubes concentriques). Des exemples numériques sont donnés.

Des recherches théoriques sont faites sur la transmission de chaleur laminaire le long des distances d'entrées hydrodynamique et thermique des conduites doubles. Pour obtenir des solutions théoriques, on a recours à l'équation intégrale non-linéaire de Volterra et à la méthode de Gauss avec intégrations numériques très précises. Les résultats théoriques coïncident très bien avec les résultats expérimentaux. De plus, les calculs théoriques des longueurs d'entrées hydrodynamique et thermique des tubes doubles sont décrits.

Zusammenfassung—Mit Hilfe von Gleichungen der Hydrodynamik, von Bessel-Funktionen und der endlichen Hankeltransformation werden Geschwindigkeitsverteilung, Druckabfall und Länge des hydrodynamischen Anlaufs bei Rohren mit ringförmigem Querschnitt (Ringraum) theoretisch bestimmt und Beispiele der numerischen Berechnung angegeben. Die theoretische Erforschung der hydrodynamischen und thermischen Anlaufstreckebei laminarem Wärmeübergang in Doppelrohren wird vervollständigt. Die theoretischen Lösungen beruhen auf der nichtlinearen Volterra-Integralgleichung und der Gauss-Methode mit ihrer hohen Genauigkeit für die numerische Integration und zeigen sehr gute Übereinstimmung mit den Versuchen. Weiterhin sind für Doppelrohre theoretische Berechnungen der Länge des hydrodynamischen und thermischen Anlaufbereichs beschrieben.

Аннотация—Теоретически методами гидродинамики, функцией Бессаля и конечного преобразования Ханкеля найдены распределение скоростей, падение давления и длина гидродинамического входного участка труб с кольцевым зазором. Приведены также примеры числовых расчётов.

Выполнены теоретические исследования теплообмена в ламинарном потоке на гидродинамическом и термическом входных участках труб с кольцевым зазором. В теоретических исследованиях использовались нелинейные интегральные уравнения Вольтерра и метод Гаусса. Последний применялся с высокой точностью при численном интегрировании. Теоретические решения хорошо согласуются с экспериментальными данными. Описаны теоретические расчёты длин гидродинамического и термического входных участков труб с кольцевым зазором.

NOTATION

a, thermal diffusivity (cm/sec);
C_r, radial velocity (cm/sec);
C_Z, axial velocity (cm/sec);
$D_2 = 2r_2$, inner diameter of outer pipe (cm);
$D_1 = 2r_1$, outer diameter of inner pipe (cm);
$F_0(r)$, inlet velocity of entry length (cm/sec);

$F_1(r)$, velocity of steady flow fully developed in the end of entry length (cm/sec);
$f_0(x)$, dimensionless velocity in the inlet;
$f_1(x)$, dimensionless velocity in the exit;
$f_0(z)$, dimensionless temperature distribution of the surface of inner pipe;
J_0, Bessel function;
L, heating length (cm);
L_V, length of hydrodynamic or velocity entry length (cm);
L_T, thermal entry length (cm);
$Nu = \alpha_1 . (D_2 - D_1)/\lambda_1$, Nusselt number;
p_0, inlet pressure (kg/cm²);
p, pressure (kg/cm²);
Pr, ν/a: Prandtl number;
$Re = \bar{w} . (D_2 - D_1)/\nu$, Reynolds number;
r, radial co-ordinate;
r_1, outer radius of inner pipe (cm);
r_2, inner radius of outer pipe (cm);
$t = (T - T_0)/(T_{w1m} - T_{w2m})$, dimensionless temperature distribution;
T_{w1m}, mean temperature of the surface of inner pipe wall (°C);
T_{w2m}, mean temperature of the surface of the outer pipe wall (°C);
T, fluid temperature (°C);
T_0, inlet temperature of fluid (°C);
u, dimensionless unknown function;
v, dimensionless unknown function;
w, dimensionless unknown function;
$\bar{w}$, mean flow velocity (cm/sec);
w_0, dimensionless velocity distribution of concentric pipes with annular space (annulus);
x, $(r - r_1)/(r_2 - r_1)$;
Y_0, Neumann function;
Z, axial co-ordinate;
z $= Z/L$;
α_1, heat transfer coefficient (kcal/m²h °C);
δ_t, thickness of thermal boundary layer (cm);
σ_2, $Re . Pr . (D_2 - D_1)/L$;
ν, kinematic viscosity (cm²/sec);
ρ, density (kgs²/cm⁴);
ψ, stream function (cm³/sec);
λ, parameter;
λ_1, thermal conductivity (kcal/m-h °C);
ζ, variable.

1. INTRODUCTION

In industrial heat exchangers and atomic reactors, there are many cases where heat transfer begins immediately at the entrance of double pipes, and therefore research of heat transfer in double pipes, in hydrodynamic entry length and in thermal entry length, is necessary, but at the moment there is little being done.

In the theoretical analysis of heat transfer of the entry length in the pipes with annular space, the velocity distribution, the pressure drop, and the hydrodynamic entry length are necessary factors, and therefore, the author determines them theoretically by hydrodynamics, Bessel functions and the finite Hankel transforms, under the given conditions of wall surface of the inner pipe and outer pipe and the inlet and outlet; examples of numerical calculations are also shown.

The theoretical researches of laminar heat transfer of double pipes in hydrodynamic entry length and in thermal entry length are given.

In the theoretical solutions, non-linear Volterra's integral equation and Gauss's method (with a high accuracy in the method of numerical integration) are used and it is shown that theoretical solutions coincide with experimental values very well. Moreover, theoretical calculations of the length of hydrodynamic entry region and the thermal entry region of double pipes are described.

2. THEORETICAL ANALYSIS OF VELOCITY DISTRIBUTION OF HYDRODYNAMIC ENTRY LENGTH

Equations of motion in hydrodynamics:

$$C_r \frac{\partial C_r}{\partial r} + C_Z . \frac{\partial C_r}{\partial Z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left(\nabla^2 C_r - \frac{C_r}{r^2}\right) \quad (1)$$

$$C_r \frac{\partial C_Z}{\partial r} + C_Z \frac{\partial C_Z}{\partial Z} = -\frac{1}{\rho}\frac{\partial p}{\partial Z} + \nu\nabla^2 C_Z \quad (2)$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial Z^2}. \tag{3}$$

Equation of continuity:

$$\frac{1}{r}\frac{\partial}{\partial r}(r \,.\, C_r) + \frac{\partial C_Z}{\partial Z} = 0. \tag{4}$$

Boundary conditions of the concentric pipes with annular space:

$$(C_r)_{r=r_1} = 0 \tag{5}$$

$$(C_r)_{r=r_2} = 0 \tag{6}$$

$$(C_r)_{Z=0} = 0 \tag{7}$$

$$(C_r)_{r=r_1} = 0 \tag{8}$$

$$(C_Z)_{r=r_2} = 0 \tag{9}$$

$$(C_Z)_{Z=0} = F_0(r) \tag{10}$$

$$(C_Z)_{Z=L} = F_1(r). \tag{11}$$

To eliminate the term of pressure p from the equations (1) and (2), operate

$$\frac{\partial}{\partial r}(2) - \frac{\partial}{\partial Z}(1),$$

and use the stream function ψ.

$$C_Z = \frac{1}{r}\frac{\partial \psi}{\partial r} \tag{12}$$

$$C_r = -\frac{1}{r}\cdot\frac{\partial \psi}{\partial Z}. \tag{13}$$

To write in dimensionless form, put

$$\psi = \bar{w}(r_2 - r_1)^2 \cdot \phi \tag{14}$$

$\bar{w}$: mean velocity of flow (cm/sec), $x = (r - r_1)/(r_2 - r_1)$, $z = Z/L$,

$$\phi = \phi_0 + \lambda\phi_1 + \lambda^2\phi_2 + \tag{15}$$

(λ = parameter),

$$\nabla_0^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{x + r_1/(r_2 - r_1)}\frac{\partial}{\partial x} + \left(\frac{r_2 - r_1}{L}\right)^2 \frac{\partial^2}{\partial Z^2} \tag{16}$$

$$\nabla_0^2\nabla_0^2\phi = \lambda \,.\, N(\phi) \tag{17}$$

$N(\phi)$ all terms except $\nabla_0^2\nabla_0^2\phi$, when the terms of p are eliminated from equations (1) and (2).

For simplification, omit the boundary conditions of (6) and (7) and equate the coefficient of parameter λ from equations (15) and (17), and then we obtain the next relations.

$$\nabla_0^2\nabla_0^2\phi_0 = 0 \tag{18}$$

$$\left(\frac{\partial \phi_0}{\partial x}\right)_{Z=0} = f_0(x) \tag{19}$$

$$\left(\frac{\partial \phi_0}{\partial x}\right)_{Z=1} = f_1(x) \tag{20}$$

$$\left(\frac{\partial \phi_0}{\partial x}\right)_{x=0} = 0 \tag{21}$$

$$\left(\frac{\partial \phi_0}{\partial x}\right)_{x=1} = 0 \tag{22}$$

$$\left(\frac{\partial \phi_0}{\partial Z}\right)_{x=0} = 0 \tag{23}$$

$$\nabla_0^2\nabla_0^2\phi_1 = F_1(x, z) \tag{24}$$

$$\left(\frac{\partial \phi_1}{\partial x}\right)_{Z=0} = 0 \tag{25}$$

$$\left(\frac{\partial \phi_1}{\partial x}\right)_{Z=1} = 0 \tag{26}$$

$$\left(\frac{\partial \phi_1}{\partial x}\right)_{x=0} = 0 \tag{27}$$

$$\left(\frac{\partial \phi_1}{\partial x}\right)_{x=1} = 0 \tag{28}$$

$$\left(\frac{\partial \phi_1}{\partial z}\right)_{x=0} = 0 \tag{29}$$

ϕ_2, ϕ_3, etc. take the same form as (24) to (29).

$f_0(x)$ and $f_1(x)$ are respectively, the dimensionless velocity distributions in the inlet ($z = 0$) and the exit ($z = 1$), and are respectively the dimensionless forms of $F_0(r)$ and $F_1(r)$ by (12) and (14).

$F_1(x, z)$ of (24) is the known function substituted ϕ_0 of (18) to (23) for ϕ in $N(\phi)$ of (17).

$$\phi_0 = u + z\frac{\partial v}{\partial z} \tag{30}$$

and we obtain the next equations.

$$\nabla_0^2 u = 0 \tag{31}$$

$$\left(\frac{\partial u}{\partial x}\right)_{Z=0} = f_0(x) \tag{32}$$

$$\left(\frac{\partial u}{\partial x}\right)_{Z=1} = \tfrac{1}{2} f_1(x) \tag{33}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=0} = 0 \tag{34}$$

$$\left(\frac{\partial u}{\partial x}\right)_{x=1} = 0 \tag{35}$$

$$\left(\frac{\partial u}{\partial z}\right)_{x=0} = 0 \tag{36}$$

$$\nabla_0^2 v = 0 \tag{37}$$

$$\left\{\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z}\right)\right\}_{Z=0} = f_0(x) \tag{38}$$

$$\left\{\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z}\right)\right\}_{Z=1} = \tfrac{1}{2} f_1(x) \tag{39}$$

$$\left\{\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z}\right)\right\}_{x=0} = 0 \tag{40}$$

$$\left\{\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial z}\right)\right\}_{x=1} = 0 \tag{41}$$

$$\left(\frac{\partial v}{\partial z}\right)_{x=0} = 0 \tag{42}$$

$$\left(\frac{\partial^2 v}{\partial z^2}\right)_{x=0} = 0. \tag{43}$$

Put $u = \int_0^x y_1 \,.\, \mathrm{d}x$ and substitute in (31) to (36).

Differentiate once both sides of (31) with respect to x, and for simplicity, omit the term of $-[x + r_1/(r_2 - r_1)]^{-2} \,.\, y_1$ and we can obtain the first approximation of y_1 from the next equations.

$$\nabla_0^2 y_1 = 0 \tag{44}$$

$$(y_1)_{Z=0} = f_0(x) \tag{45}$$

$$(y_1)_{Z=1} = \tfrac{1}{2} f_1(x) \tag{46}$$

$$(y_1)_{x=0} = 0 \tag{47}$$

$$(y_1)_{x=1} = 0. \tag{48}$$

Next, put $v = \int_0^x \int_0^z y_2 \,.\, \mathrm{d}z \,.\, \mathrm{d}x$, substitute into (37) to (43), differentiate one time both sides of (37) with respect to x and z respectively, for simplicity, omit the term of

$$-[x + r_1/(r_2 - r_1)]^{-2} \,.\, y_2,$$

and we can obtain the first approximation of y_2 from the next equations.

$$\nabla_0^2 y_2 = 0 \tag{49}$$

$$(y_2)_{z=0} = f_0(x) \tag{50}$$

$$(y_2)_{z=1} = \tfrac{1}{2} f_1(x) \tag{51}$$

$$(y_2)_{x=0} = 0 \tag{52}$$

$$(y_2)_{x=1} = 0. \tag{53}$$

Substitute y_1 and y_2 into (12), (14) and (30), and we obtain the next equation as the first approximation of $C_Z/\bar{w}$ which satisfies boundary conditions completely in the place closest to the wall of the inner pipe.

$$\left.\begin{aligned} \frac{C_Z}{\bar{w}} &= \frac{(1+z)}{[x + r_1/(r_2 - r_1)]} \\ &\sum_{s=1}^{\infty} \frac{2 \,.\, U_0(x, S)}{\sinh k_s\,[L/(r_2 - r_1)]\,\Big[[r_2/(r_2 - r_1)]^2 \big(U_0'\{k_s\,[r_2/(r_2 - r_1)]\}\big)^2 - [r_1/(r_2 - r_1)]^2 \,.\, \big(U_0'\{k_s[r_1/(r_2 - {}_1)]\}\big)^2\Big]} \\ &\times \{\sinh k_s\,[L/(r_2 - r_1)]\,(1 - z) \,.\, \textstyle\int_0^1 f_0(t) \,.\, U_0(t, S)t \,.\, \mathrm{d}t \\ &\qquad + \tfrac{1}{2} \sinh k_s\,[L/(r_2 - r_1)]\, z \textstyle\int_0^1 f_1(t) \,.\, U_0(t, S)t \,.\, \mathrm{d}t\} \end{aligned}\right\} \tag{54}$$

$$U_0(x, s) = \frac{J_0\{k_s[x + r_1/(r_2 - r_1)]\}}{J_0\{k_s\, r_1/(r_2 - r_1)\}} - \frac{Y_0\{k_s[x + r_1/(r_2 - r_1)]\}}{Y_0\{k_s\, r_1/(r_2 - r_1)\}} \tag{55}$$

k_s is a root of the next equation.

$$J_0\left\{k\,\frac{r_2}{r_2 - r_1}\right\} \,.\, Y_0\left\{k\,\frac{r_1}{r_2 - r_1}\right\} = Y_0\left\{k\,\frac{r_2}{r_2 - r_1}\right\} \,.\, J_0\left\{k\,\frac{r_1}{r_2 - r_1}\right\} \tag{56}$$

3. AN EXAMPLE OF NUMERICAL CALCULATION OF DISTRIBUTION OF VELOCITY IN THE HYDRODYNAMIC ENTRY LENGTH

In the numerical calculation of velocity distribution in the hydrodynamic entry length from (54), we must give $f_0(t)$ of the inlet velocity distribution and $f_1(t)$ of the velocity distribution of steady flow in the down stream passed over the hydrodynamic entry length and fully developed.

The inlet velocity distribution is generally $F_0(r)$ of the function of radius as (10), and considered to be the distribution of a rectangle with corners slightly rounded. When we represent $f_0(t)$ in this curve, we must use the elliptic functions of the first and second kind, and therefore, here, for simplicity, $f_0(t)$ is treated as the uniform distribution, that is to say,

$$(Cz)_{Z=0} = \bar{w}. \tag{57}$$

Therefore

$$f_0(t) = \left(t + \frac{r_1}{r_2 - r_1}\right). \tag{58}$$

In the next, velocity distribution of steady flow in the down stream fully developed is shown by the next equation.

$$(Cz/\bar{w})_{Z=L} = 2 \,.\, w_0(x). \tag{59}$$

Therefore

$$f_1(t) = 2 \,.\, w_0(t) \times \left(t + \frac{r_1}{r_2 - r_1}\right) \tag{60}$$

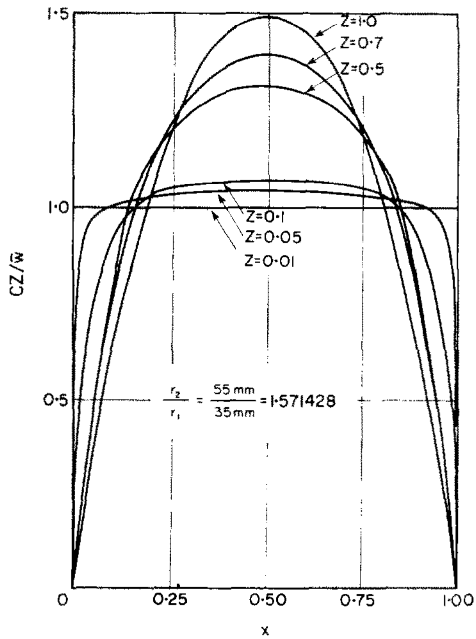

FIG. 1. Velocity distribution in hydrodynamic entry length of double pipes.

$$f_1(t) = 2\,\frac{[-\{(r_2/r_1 - 1)t + 1\}^2 + 1 + \{(r_2/r_1)^2 - 1\}/\log_e r_2/r_1 \times \log_e \{(r_2/r_1 - 1)t + 1\}]}{(r_2/r_1)^2 + 1 - \{(r_2/r_1)^2 - 1\}/\log_e r_2/r_1} \times \left(t + \frac{r_1}{r_2 - r_1}\right). \tag{61}$$

To simplify (54), take the first term of sinh and substitute $f_0(t)$ and $f_1(t)$ mentioned above, and we obtain the next equation as the approximate equation for the numerical calculation.

$$\frac{Cz}{\bar{w}} \doteqdot (1 - z^2) + (z + z^2) \,.\, w_0(x). \tag{62}$$

This equation is the case taken only the first term of sinh and considered simply like (59), and therefore, this equation does not satisfy the boundary conditions at $x = 0$ and $x = 1$ only, but it is found to be $(Cz/\bar{w})_{x=0,\, x=1} = 0$ from (54) which is a more accurate solution than (62), and so (62) becomes an approximate equation which is very convenient for the calculation of velocity distribution at the arbitrary points except for $x = 0$ and $x = 1$ only, i.e. the velocity of $x = 0$ and $x = 1$ give respectively the velocity ($=0$) on the surface of inner and outer pipe-walls.

In the case of $r_2/r_1 = d_2/d_1 = 55$ mm/35 mm $=$ 1·571428 in (62) and (61), an example with the given value of z is shown by Fig. 1. In Fig. 1, the velocity distribution at the vicinity of the inlet is nearly uniform, but the velocity distribution down stream (in the direction of the outlet) becomes parabolical and is not symmetrical with respect to the centre ($x = 0{\cdot}5$) of the annulus; the velocity at the side of the inner pipe ($x = 0$) is larger than the velocity at the side of the outer pipe ($x = 1$).

The velocity distribution of hydrodynamic entry length of double pipes is a complicated function of r_2/r_1 ratio of radius and of radial and axial direction; by (54) or (62) velocity of arbitrary point is also calculable in the case of the ratio of the radius r_2/r_1 given arbitrarily.

4. AN EXAMPLE OF NUMERICAL CALCULATION OF PRESSURE DROP

Integrate the pressure drop [1]

$$-\frac{1}{\rho}\frac{\partial p}{\partial Z} = (C_Z)_{x=0\cdot5} \times \frac{\partial (C_Z)_{x=0\cdot5}}{\partial Z} \quad (63)$$

and write in dimensionless form. In Fig. 2

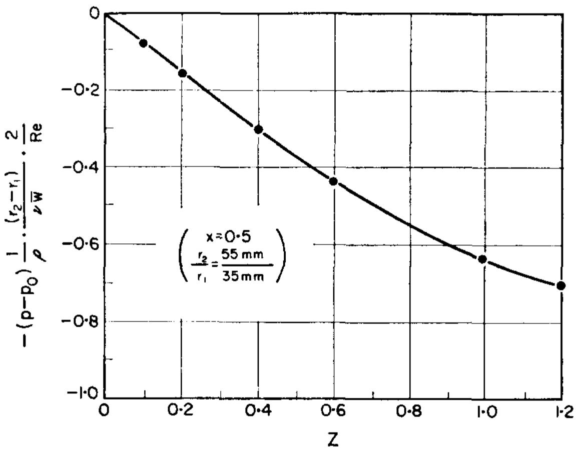

FIG. 2. Pressure drop in hydrodynamic entry length of double pipes.

($r_2/r_1 = 55$ mm/35 mm, $x = 0{\cdot}5$), z is abscissa, and the ordinate is

$$-(p - p_0)1/\rho \,.\, (r_2 - r_1)\nu\bar{w} \times 2/Re,$$

where $Re = \bar{w} \,.\, 2(r_2 - r_1)/\nu$, p_0 = pressure at inlet (kg/cm²).

Then, put $w = C_Z/\bar{w}$ and substitute (62) for w and we obtain (64).

$$\left.\begin{aligned} &-(p - p_0)\frac{1}{\rho}\,\frac{(r_2 - r_1)}{\nu\,.\,\bar{w}}\cdot\frac{2}{Re} = \\ &\int_0^Z \left(w\,.\,\frac{\mathrm{d}w}{\mathrm{d}z}\right)_{x=0\cdot5}.\,\mathrm{d}z \approx -z^2 + \tfrac{1}{2}z^4 \\ &+ w_{0(x=0\cdot5)} \times (z + z^2 - z^3 - z^4) \\ &+ \{w_{0(x=0\cdot5)}\}^2 \cdot (\tfrac{1}{2}z^2 + z^3 + \tfrac{1}{2}z^4) \end{aligned}\right\} . (64)$$

We can find from (64) that the pressure drop in the hydrodynamic entry length is shown by the biquadratic equation of z.

5. L_V: HYDRODYNAMIC ENTRY LENGTH

By the above-mentioned, the velocity distribution and the pressure drop in the hydrodynamic entry length are obtained in the dimensionless forms, and so, next, the hydrodynamic entry length must be determined.

In (1), (2), (4), assume $C_r \approx 0$ simply, put $w = C_Z/\bar{w}$ and we obtain the next equations in the dimensionless form from (2).

$$\frac{\partial^2 w}{\partial x^2} + \frac{1}{x + r_1/(r_2 - r_1)}\,\frac{\partial w}{\partial x} + \left(\frac{r_1 - r_2}{L}\right)^2 \frac{\partial^2 w}{\partial z^2} = F(x, z) \quad (65)$$

$$(w)_{x=0} = 0 \quad (66)$$

$$(w)_{x=1} = 0 \quad (67)$$

$$(w)_{z=0} = f(x) \quad (68)$$

$$(\partial w/\partial z)_{z=0} = 0 \quad (69)$$

(put $C_r \approx 0$ in the equation of continuity)

$F(x, z)$ = the term of pressure drop =

$$\frac{1}{\rho}\,\frac{\partial p}{\partial Z}\,\frac{(r_2 - r_1)^2}{\nu\,.\,\bar{w}} = -\frac{Re}{2}\left(\frac{r_2 - r_1}{L}\right) w_1 \frac{\partial w_1}{\partial Z} \quad (70)$$

(x) in (68) is the inlet velocity in the dimensionless form.

If we assume $F(x, z)$ to be substituted (62) for w_1 of (70) approximately, we can treat $F(x, z)$ as a known function.

By the boundary conditions (66) and (67), introduce the finite Hankel transform [2] to equation (65) and put

$$H_0[w] = u = \int_0^1 \left(x + \frac{r_1}{r_2 - r_1}\right) . w . \left[J_0\left\{k_s\left(x + \frac{r_1}{r_2 - r_1}\right)\right\} . Y_0\left\{k_s \frac{r_2}{r_2 - r_1}\right\} - J_0\left\{k_s \frac{r_2}{r_2 - r_1}\right\} . Y_0\left\{k_s\left(x + \frac{r_1}{r_2 - r_1}\right)\right\}\right] . dx \quad (71)$$

$$H_0[F] = \phi, \qquad H_0[f(x)] = \psi$$

and we obtain the next equations.

$$-k_s^2 . u + \left(\frac{r_2 - r_1}{L}\right)^2 . \frac{d^2u}{dz^2} = \phi(z) \quad (72)$$

$$(u)_{z=0} = \psi \quad (73)$$

$$(du/dz)_{z=0} = 0 \quad (74)$$

k_s is a root of the next transcendental equation.

$$J_0\left\{k_s\left(\frac{r_1}{r_2 - r_1}\right)\right\} . Y_0\left\{k_s\left(\frac{r_2}{r_2 - r_1}\right)\right\} - J_0\left\{k_s\left(\frac{r_2}{r_2 - r_1}\right)\right\} . Y_0\left\{k_s\left(\frac{r_1}{r_2 - r_1}\right)\right\} = 0. \quad (75)$$

If we apply Laplace transform to (72) by (73) and (74), we can find u by the inversion theorem.

$$u = \psi . \cosh k_s \frac{L}{r_2 - r_1} z + \frac{1}{k_s} \frac{L}{r_2 - r_1} \int_0^z \phi(\zeta) \sinh k_s \frac{L}{r_2 - r_1} (z - \zeta) . d\zeta. \quad (76)$$

By the inversion theorem of the finite Hankel transform, the velocity distribution $w(x, z)$ is found in the next equation.

$$w(x, z) = \sum_{s=1}^{\infty} \frac{2 . k_s^2 . J_0^2\{k_s r_1/(r_2 - r_1)\} . (J_0\{k_s [x + r_1/(r_2 - r_1)]\} . Y_0\{k_s r_2/(r_2 - r_1)\} - J_0\{k_s r_2/(r_2 - r_1)\} . Y_0\{k_s [x + r_1/(r_2 - r_1)]\})}{J_0^2\{k_s r_2/(r_2 - r_1)\} - J_0^2\{k_s r_1/(r_2 - r_1)\}}$$
$$\times \Big[\cosh k_s L/(r_2 - r_1) z \int_0^1 [x + r_1/(r_2 - r_1)] f(x) (J_0\{k_s [x + r_1/(r_2 - r_1)]\} . Y_0\{k_s r_2/(r_2 - r_1)\} - J_0\{k_s r_2/(r_2 - r_1)\} . Y_0\{k_s [x + r_1/(r_2 - r_1)]\}) dx$$
$$+ (1/k_s) . L/(r_2 - r_1) \int_0^z \{\int_0^1 [x + r_1/(r_2 - r_1)] . F(x, \zeta) . (J_0\{k_s [x + r_1/(r_2 - r_1)]\} \, Y_0\{k_s r_2/(r_2 - r_1)\} - J_0\{k_s r_2/(r_2 - r_1)\} \cdot Y_0\{k_s [x + r_1/(r_2 - r_1)]\}) \cdot dx\} . \sinh k_s L/(r_2 - r_1) (z - \zeta) . d\zeta\Big] \quad (77)$$

In (77), $w(x, z = 1)$ in the case of $z = 1$ is the velocity distribution in the end of the hydrodynamic entry length.

To find $L/(r_2 - r_1)$ from the condition that $w(x, z = 1)$ must to be equal to $2w_0(x)$, the steady velocity distribution in the down stream fully developed, if we apply the finite Hankel transform to $2 . w_0(x)$ and next, by the inversion theorem, expand $2w_0(x)$ by J_0 and Y_0, we obtain an equation from (61) and (77).

In that equation, $L/(r_2 - r_1)$ is contained as an unknown number and so, from the relation, if we find $L/(r_2 - r_1)$, $L = L_V$ the length of the hydrodynamic entry length can be determined. From this, it is found that $L/(r_2 - r_1)$ is a function of Reynolds number Re and the ratio of radius r_2/r_1.

Therefore, by the numerical solution of Graeffe's method as the method of solution of equation of higher order about L in the case of giving Re, r_2 and r_1, it is possible to find L more accurately, but here, the next approximate calculation is performed simply.

(i) *The case of small Re*

We apply Simpson's $\frac{1}{3}$ rule to the approximate calculation of definite integral $\int_0^{z=1}(\zeta) . d\zeta$ with respect to ζ in (77) and next, in the same way, we apply also Simpson's $\frac{1}{3}$ rule to the calculation of $\int_0^1(x) . dx$ with respect to x in (77) and, neglecting the little terms, we obtain the next formulae.

$$\frac{L}{r_2 - r_1} = \sum_{s=1}^{\infty} \frac{1}{k_s} . \log_e \sqrt{\{A + \sqrt{(A^2 - 1)}\}} \quad (78)$$

$$A = \frac{2 \, . \, w_{0(x=0.5)}}{1 - Re/(6 \cdot k_s) \, . \, w_{0(x=0.5)}}. \quad (79)$$

The limit of application of (78) is shown in the next equation.

$$[1 - Re/(6 \cdot k_s) \, . \, w_{0(x=0.5)} > 0]. \quad (80)$$

(ii) *The case of large Re*

(We can know the standard of the variation of *Re* by equation (80).)

Calculate exactly $\int_0^1 d\zeta$ with respect to ζ of (77) by the formula of integration, apply Simpson's $\frac{1}{3}$ rule to $\int_0^1 dx$ with respect to x and, neglecting the little terms, we obtain the next formula.

$$\frac{L}{r_2 - r_1} = Re \, . \, w_{0(x=0.5)} \sum_{s=1}^{\infty} \frac{1}{k_s^2}. \quad (81)$$

From (78) and (81), we can find that $L(=L_V)$ the length of the hydrodynamic entry length is the function of Reynolds number *Re* and ratio of radius r_2/r_1.

6. AN EXAMPLE OF NUMERICAL CALCULATION OF $(L = L_V)$ THE HYDRODYNAMIC ENTRY LENGTH

(i) $Re = 10$

When $r_2/r_1 = 55/35 = 1{\cdot}571428$, we obtain the next from (78).

$$L/(r_2 - r_1) = 0{\cdot}250150 \text{ (first term)}. \quad (82)$$

When

$$r_2/r_1 = 1{\cdot}2, \quad L/(r_2 - r_1) = 0{\cdot}2489664 \text{ (first term)}. \quad (83)$$

Next, when $r_2/r_1 = 55/33 = 1{\cdot}571428$, calculate to the third term ($s = 3$) and we obtain

$$L/(r_2 - r_1) = 0{\cdot}4098550 \text{ (to } s = 3). \quad (84)$$

(ii) *The case of large Re (calculated from equation (81))*

When $r_2/r_1 = 55/33 = 1{\cdot}571428$, calculate to the third term ($s = 3$) and we obtain

$$L/(r_2 - r_1) = 0{\cdot}1039973 \, . \, Re. \quad (85)$$

When $r_2/r_1 = 1{\cdot}2$, calculate to the third term ($s = 3$) and we obtain

$$L/(r_2 - r_1) = 0{\cdot}1035215 \, . \, Re. \quad (86)$$

Equations (84) and (85) are shown by Fig. 3 in log scale taking *Re* as abscissa. By these examples of numerical calculation, the difference of the values of equations (82) and (83) is about 0·0011 and the difference of the coefficients of *Re* in equations (85) and (86) is 0·00047.

Within the limits of radius ratio

$$(1{\cdot}01 \leqslant r_2/r_1 \leqslant 1{\cdot}6)$$

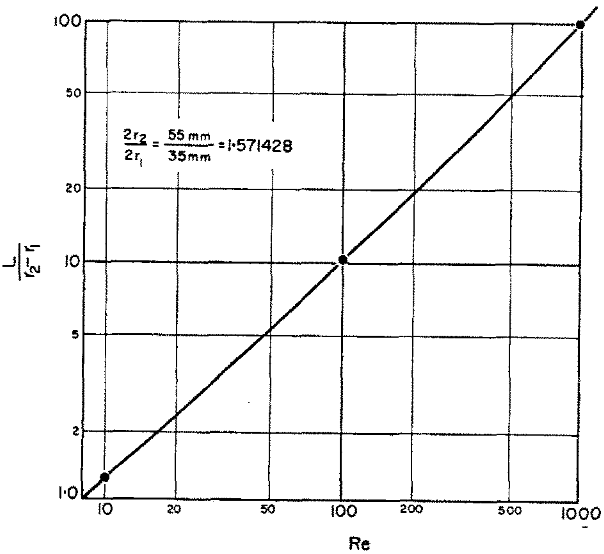

FIG. 3. Relation of the length of hydrodynamic entry length and *Re*.

the influence of radius ratio r_2/r_1 is small and the influence of Re is large, as Fig. 3.

7. HEAT TRANSFER OF HYDRODYNAMIC ENTRY LENGTH

To find the approximate solution simply by the theory of boundary layer, put

$$t = (T - T_0)/(T_{w1m} - T_{w2m}),$$
$$x = (r - r_1)/(r_2 - r_1),$$
$$z = Z/L,$$
$$\sigma_2 = Re \,.\, Pr \,.\, (D_2 - D_1)/L$$

and we obtain the next energy equation of the boundary layer.

$$\frac{\sigma_2}{4} \cdot \frac{\partial}{\partial z} \int_0^{\delta t/(r_2 - r_1)} w(x, z) \,.\, t(x, z) \,.\, dx = - \left(\frac{\partial t}{\partial x}\right)_{x=0}. \quad (87)$$

From (62),

$$w(x, z) \doteqdot (1 - z^2) + (z + z^2) \,.\, w_0(x) \doteqdot 1 + z \,.\, w_0(x).$$

(When $z \doteqdot 0$.)

Put

$$t \doteqdot f_0(z) \left\{ 1 - 2 \left(\frac{\delta_t}{r_2 - r_1}\right)^{-1} . x + \left(\frac{\delta_t}{r_2 - r_1}\right)^{-2} . x \right\},$$

$f_0(z)$ = temperature distribution of the surface of inner pipe.

Substitute w and t mentioned above for (87), and we are able to solve reducing to non-linear Volterra's integral equation.

The first approximation in the case of constant temperature of inner pipe wall is

$$\frac{\delta_t}{(r_2 - r_1)} \doteqdot \frac{(48)^{1/2}}{(\sigma_2)^{1/2}} \cdot z^{1/2}. \quad (88)$$

The Nusselt number of the inner pipe in this case is the next.

$$Nu_1 = \frac{a_1 \,.\, (D_2 - D_1)}{\lambda_1} = - 2 \frac{\int_0^1 (\partial t/\partial x)_{x=0} \,.\, dz}{\int_0^1 f_0(z) \,.\, dz} \doteqdot (\tfrac{2}{3})^{1/2} \,.\, \sigma_2^{1/2} \doteqdot \doteqdot 0{\cdot}8165 \,.\, \sigma_2^{0\cdot 5}. \quad (89)$$

Gauss's method is used with high accuracy in the numerical integration. Therefore by equation (89), we can calculate simply the Nusselt number of laminar flow through hydrodynamic entry length in the case of constant temperature of inner pipe wall.

In the next, when we consider the arbitrary temperature distribution $f_0(z)$ of inner pipe wall, in the similar way, we can obtain the next equation.

$$Nu_1 \doteqdot \sigma_2^{1/2} \,.\, (\tfrac{2}{3})^{1/2} \,.\, \left[1 + \frac{1}{f_{0(z=0\cdot5)}} \left\{\frac{\partial f_0(z)}{\partial z}\right\}_{z=0\cdot5}\right]^{1/2} \quad (90)$$

In Fig. 4 (both log scale), the experimental results [3] of water heated from the inner pipe (mark ○ shows the data of vertical type, height is 3 m; mark × shows the data of horizontal type, length is 5 m) are shown, in the case of laminar flow only except for the influence of free convection.

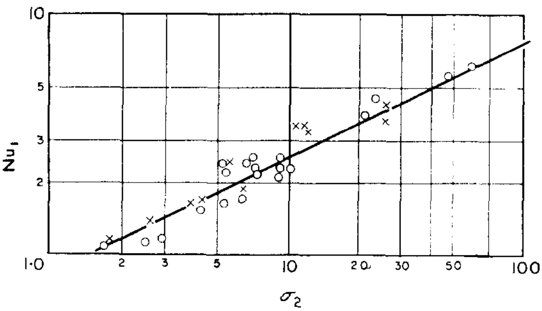

FIG. 4. Heat transfer of hydrodynamic entry length.

Moreover, the solid line in Fig. 4 shows the results calculated by (89). We can observe that theoretical results coincide with experimental results very well.

8. L_T: THE THERMAL ENTRY LENGTH

To find L_T by the purely theoretical calculations, applying theory of boundary layer, as the temperature distribution in the thermal boundary layer of thermal entry length, put

$$t = f_0(z) \,.\, (\tfrac{1}{2}x^3 - 3 \,.\, \tfrac{1}{2}x + 1),$$

and as the velocity distribution $2 \,.\, w_0(x)$, put

$$2 \, . \, w_0(x) = 2 \, \frac{[-\{(r_2/r_1 - 1)x + 1\}^2 + 1 + \{(r_2/r_1)^2 - 1\}/\log_e (r_2/r_1) \times \log_e \{(r_2/r_1 - 1)\, x + 1\}]}{(r_2/r_1)^2 + 1 - \{(r_2/r_1)^2 - 1\}/\log_e r_2/r_1}.$$

From the energy equation of the boundary layer

$$\frac{\sigma_2}{4} \int_0^1 \frac{\partial t}{\partial z} \, . \, 2 \, . \, w_0(x) \, . \, dx = - \left(\frac{\partial t}{\partial x}\right)_{x=0} \qquad (91)$$

we obtain the next relation (92),

$$L_T/(r_2 - r_1) = L_V/(r_2 - r_1) + z \, . \, L/(r_2 - r_1) \quad (92)$$

where L_T is the thermal entry length and L_V is the hydrodynamic (or velocity) entry length.

$$L_V/(r_2 - r_1) = Re \, . \, w_0(x = 0{\cdot}5) \sum_{s=1}^{\infty} 1/k_s^2.$$

In general case to find z from (91), if we give $f_0(z)$ as the polynomial of higher order of z, we can solve algebraic equation of higher order of z by the numerical method of solution of Graeffe's method, but here, to find simply by the experimental results of water heated from the inner pipe, put

$$f_0(z) \simeq - 0{\cdot}7167z^2 + 1{\cdot}2683z + 0{\cdot}045.$$

The results of one example in this case are shown by the next value of z.

$$z = z(\sigma_2) \simeq \sigma_2 \times w_{0(x=0\cdot5)} \times 0{\cdot}069444 + 0{\cdot}88482 + [\{\sigma_2 \, . \, w_{0(x=0\cdot5)} \times 0{\cdot}00087332 + 0{\cdot}011127\}^2 + \{0{\cdot}062788 - \sigma_2 \times w_{0(x=0\cdot5)} \times 0{\cdot}12289\}]^{1/2}. \qquad (93)$$

From (93), we can understand that $z(\sigma_2)$ is the function of w_0, i.e. $r_2/r_1 = D_2/D_1 =$ diameter ratio and $\sigma_2 = Re \, . \, Pr \, . \, (D_2 - D_1)/L$.

As we can calculate L_T, the thermal entry length in the case of water from equation (92) and (93), in Fig. 5 (in both log scale), the results of theoretical calculation in the case of water heated from the inner pipe are shown by the relations of $L_T/(r_2 - r_1)$ and $L_V/(r_2 - r_1)$ versus Re.

We can understand that L_T, the thermal entry length of laminar flow only is fairly long. L_T and L_V are the results of theoretical calculations only. The influence of Pr upon L_T and the thermal entry length calculable by (91) and for example if we calculate the case of air it is known that L_T is shorter than the case of water, and these agree with common sense.

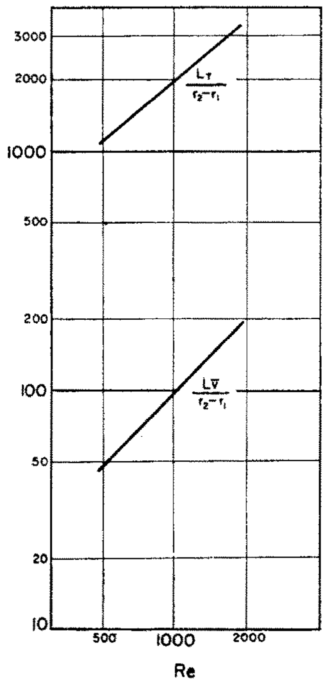

FIG. 5. An example of numerical calculation of L_T and L_V.

9. HEAT TRANSFER OF THE THERMAL ENTRY LENGTH

Put

$$t \simeq f_0(z) \left\{1 - 2 \left(\frac{\delta_t}{r_2 - r_1}\right)^{-1} . \, x + \left(\frac{\delta_t}{r_1 - r_2}\right)^{-2} . \, x^2\right\},$$

for temperature distribution passed over the hydrodynamic entry length and put

$$v(x) \simeq 2 \, . \, w_0(x) \simeq 8 \, . \, w_{0(x=0\cdot5)} \, . \, (x - x^2)$$

for velocity distribution and substitute the energy equation of boundary layer

$$\frac{\sigma_2}{4}\int_0^{\delta_t(r_2-r_1)}\left[\frac{\partial t}{\partial z}\times 2\,.\,w_0(x)\right].\,\mathrm{d}x = -\left(\frac{\partial t}{\partial x}\right)_{x=0}$$

and put

$$\left(\frac{\delta_t}{r_2-r_1}\right)^3 = u.$$

If we reduce to non-linear Volterra's integral equation and use the method of numerical integration by Gauss's method with high accuracy, the result of constant temperature of inner pipe wall (the first approximation) in the case of large *Pr* of water and oil etc. is as follows.

$$Nu_1 = \frac{2}{9^{1/3}}\left\{w_0(x=0\cdot 5)\right\}^{1/3}.\,\sigma_2^{1/3} + \frac{1}{6^{1/2}}\,.\,\sigma_2^{1/2} \quad (94)$$

From (94), we can understand that the Nusselt number Nu_1 is the function of w_0, i.e. $r_2/r_1 = D_2/D_1$ = diameter ratio and $\sigma_2 = Re\,.\,Pr\,.\,(D_2 - D_1)/L$.

The theoretical result calculated by (94) is shown by the solid line of Fig. 6. The experimental results [3] of laminar heat transfer only of water heated from inner pipe, neglecting the influence of free convection, is shown by Fig. 6

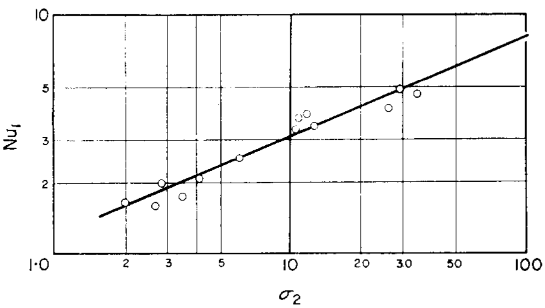

FIG. 6. Heat transfer of thermal entry length.

(in both log scales). We can understand that the experimental results coincide with the theoretical result.

In the next, if we consider $f_0(z)$, the temperature distribution of the inner pipe wall, in the same way, we obtain the next equation.

$$Nu_1 = \frac{2\{[\sigma_2\,.\,w_{0(x=0\cdot 5)}/18]\}^{1/3}\int_0^1(1/z^{1/3})f_0(z)\,.\,\mathrm{d}z}{\int_0^1 f_0(z)\,.\,\mathrm{d}z} + \frac{\sigma_2^{1/2}}{6^{1/2}}\left[1+\frac{1}{f_{0(z=0\cdot 5)}}\cdot\left\{\frac{\partial f_0(z)}{\partial z}\right\}_{z=0\cdot 5}\right]^{1\cdot 2}. \quad (95)$$

10. CONCLUSION

The author deduced (54) the velocity distribution of the hydrodynamic entry length of double pipes with annular space (annulus) which satisfies the boundary conditions of inner pipe wall, inlet and outlet (the end of the length of hydrodynamic entry length).

If we consider uniform velocity distribution at the inlet, we can use (61) and (62) as the approximate formulae. As the results of examples of numerical calculations, the velocity distributions near the inlet are almost uniform; but the velocity distributions near the down stream are parabolic, and not symmetrical with respect to the centre of the clearance of double pipes (annuli), but a bit larger at the inner pipe side of double pipes.

The pressure drop in the hydrodynamic entry length as in (64) is shown by the biquadratic equation of z (the distance from the inlet).

The determination of the hydrodynamic entry length is a very important and difficult problem; the author's solution, introduces the finite Hankel transform. According to Sneddon, the finite Hankel transform is the quickest and easiest solution in comparison with the other methods of Laplace transform and L^2-transform etc. The finite Hankel transform appears to be a very useful, convenient new method in the boundary value problems, treating especially circular rods and pipes, in heat conduction, heat transfer, hydrodynamics, elasticity and vibration etc.

When we calculate the hydrodynamic entry length, giving radius ratio r_2/r_1 and Reynolds number *Re*, we can find the accurate value for any size, by solving the algebraic equation of high order by the method of numerical solution, for example, by the Graeffe's method etc., and we can also find it simply from equations (78) and (81). The hydrodynamic entry length is shown by the function of Reynolds number *Re* and radius ratio r_2/r_1, and by the examples of numerical calculations, it is found that the influence of Reynolds number *Re* is larger than the influence of radius ratio r_2/r_1.

The author analysed theoretically the velocity distribution of the hydrodynamic entry length of double pipes with annular space (annulus), pressure drop and hydrodynamic entry length, from equations and hydrodynamics, and also showed the examples of numerical calculations. Giving dimensions arbitrarily, we can find the necessary values by the calculations mentioned above.

In the industrial heat exchangers, there are many cases where heat transfer begins instantly at the entrances of double pipes and the temperature distributes over the walls of double pipes. In these cases, the author accomplished the theoretical calculations at the heat transfer of hydrodynamic and thermal entry length. It became clear that the results of theoretical calculations coincide with the experimental results very well.

In the theoretical solutions, non-linear Volterra's integral equation and Gauss's method with a high accuracy, in the method of numerical integration, are used. In industry, there are many cases of heat transfer of both hydrodynamic entry length and a part of thermal entry length passed over hydrodynamic entry length, but research in this case being sparse, the author also accomplished the research in this case.

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